Galilean invariance and vertex renormalization in turbulence theory

W. D. McComb*
School of Physics, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom
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The Navier-Stokes equation is invariant under Galilean transformation of the instantaneous velocity field. However, the total velocity transformation is effected by transformation of the mean velocity alone. For a constant mean velocity, the equation of motion for the fluctuating velocity is automatically Galilean invariant in the comoving frame, and vertex renormalization is not constrained by this symmetry.

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Galilean invariance is often invoked for macroscopic nonlinear field problems in order to restrict the nature of possible solutions [1–12]. Such approaches have been influenced by the use of Lorentz invariance in quantum field theory. Indeed, Galilean invariance has been used as the justification of Ward identities, which in turn lead to the conclusion that in the perturbative renormalization group (RG) the vertex is not renormalized [1,9]. If correct, this conclusion is important, as it is well known that the elementary perturbation theory of the Navier-Stokes (and similar equations) generates vertex corrections.

However, the application of field-theoretic approaches to classical problems requires a certain amount of caution. It is not correct to regard the Galilean transformation as merely the low-speed form of the Lorentz transformation. In the former case, the need additionally to transform the boundary conditions may render the symmetry essentially trivial for the dynamics of the system [13], while in some classical systems the symmetry may be hidden [14], meaning that it is satisfied automatically by the formalism and hence can impose no constraint on the form of solutions. In this Brief Report we shall argue that this is the case for stochastic classical nonlinear systems where one considers fluctuations about a *mean which itself satisfies the requirement of Galilean invariance*.

As our main point is quite subtle, and our conclusion likely to prove controversial, we will present our arguments in a detailed, almost pedagogic fashion. Consider the Navier-Stokes equation (NSE) in an inertial frame S:

$$\frac{\partial V_i}{\partial t} + V_j\frac{\partial V_i}{\partial x_j} = -\frac{\partial \Pi}{\partial x_i} + \nu_0\nabla^2 V_i, \tag{1}$$

where ν_0 is the kinematic viscosity of the fluid, $V_i(\mathbf{x},t)$ and $\Pi(\mathbf{x},t)$ are the instantaneous values of the velocity and pressure, and the continuity equation takes the form $\partial V_i/\partial x_i=0$, for an incompressible fluid. For convenience we work in a system of units where the fluid density is unity.

Also, as is well known, taking the divergence of each term in (1), and invoking incompressibility, leads to a Poisson-type equation for the pressure, thus:

$$\nabla^2\Pi = -\frac{\partial(V_iV_j)}{\partial x_i\partial x_j}. \tag{2}$$

This result is used to establish the Galilean transformation of the fluid pressure.

We wish to show that the NSE takes the same form in another inertial frame $\tilde{S}$, provided all variables are replaced by variables with tildes, which are the corresponding quantities measured in $\tilde{S}$. This property is known as Galilean invariance. We begin by defining the Galilean transformation and establishing the basic transformation laws.

Consider a second inertial frame $\tilde{S}$, with constant velocity c_i and situated at $x_i=c_it$ in S at time t. Without loss of generality, we take the axes of $\tilde{S}$ to be oriented along the axes of S, and to have been coincident at $t=0$. We also make the usual assumption in Galilean relativity that time is universal and hence $t=\tilde{t}$. Then the position coordinates of an event, which are given by $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in the two coordinate frames, are related by simple vector addition, as are the velocities, thus:

$$\mathbf{x} = \mathbf{c}t + \tilde{\mathbf{x}}, \quad V_i = c_i + \tilde{V}_i. \tag{3}$$

It is easily shown that the relevant differential coefficients transform as

$$\frac{\partial \tilde{V}_i}{\partial x_j} = \frac{\partial \tilde{V}_i}{\partial \tilde{x}_j}, \quad \frac{\partial \tilde{V}_i}{\partial t} = \frac{\partial \tilde{V}_i}{\partial \tilde{t}} - c_j\frac{\partial \tilde{V}_i}{\partial \tilde{x}_j}, \tag{4}$$

and these results apply to all functions of $\mathbf{x}$ and t.

We can now transform the NSE, as given by Eq. (1) in frame S, into frame $\tilde{S}$. From Eqs. (2)–(4) it follows that $\Pi=\tilde{\Pi}$. With the substitution of this result along with Eqs. (3) and (4), the NSE in S becomes

$$\frac{\partial \tilde{V}_i}{\partial \tilde{t}} \underbrace{- c_j\frac{\partial \tilde{V}_i}{\partial \tilde{x}_j} + c_j\frac{\partial \tilde{V}_i}{\partial \tilde{x}_j}} + \tilde{V}_j\frac{\partial \tilde{V}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{\Pi}}{\partial \tilde{x}_i} + \nu_0\tilde{\nabla}^2\tilde{V}_i. \tag{5}$$

Canceling the two terms joined by the underbrace, we have Eq. (5) as the NSE in $\tilde{S}$. This establishes the Galilean invariance of the Navier-Stokes equation, as required. It should be noted that the invariance of the continuity relation follows quite trivially, from Eqs. (3) and (4).

In order to study fluctuating flows, we make the usual Reynolds decomposition which divides the instantaneous ve-

*Electronic address: w.d.mccomb@ed.ac.uk

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locity into a mean velocity and a fluctuation about that mean. We set $V_i = U_i + u_i$, where $U_i = \langle V_i \rangle$ is the mean velocity and u_i is the fluctuation about the mean. By definition, $\langle u_i \rangle = 0$, where the angle brackets $\langle \cdots \rangle$ denote an ensemble average. Similarly, we may introduce an analogous decomposition for the pressure, $\Pi = P + p$, where P is the mean pressure and the fluctuating pressure p satisfies $\langle p \rangle = 0$.

We note that these are instantaneous velocities at a specific point $\mathcal{P}$ in space-time. Hence, in terms of the two sets of coordinates of that point, we have

$$V_i(\mathbf{x},t) = c_i + \tilde{V}_i(\tilde{\mathbf{x}},\tilde{t}). \tag{6}$$

(Recall that time is universal and so strictly $\tilde{t} = t$. But we shall leave it as $\tilde{t}$ for consistency at this stage.) Applying the Reynolds decomposition to this form gives

$$U_i(\mathbf{x},t) + u_i(\mathbf{x},t) = c_i + \tilde{U}_i(\tilde{\mathbf{x}},\tilde{t}) + \tilde{u}_i(\tilde{\mathbf{x}},\tilde{t}), \tag{7}$$

and taking averages, using $\langle u_i \rangle = \langle \tilde{u}_i \rangle = 0$ and $\langle \mathbf{c} \rangle = \mathbf{c}$, yields the transformation of the mean velocity as

$$U_i(\mathbf{x},t) = c_i + \tilde{U}_i(\tilde{\mathbf{x}},\tilde{t}). \tag{8}$$

Then, subtracting this result from Eq. (7), we obtain

$$u_i(\mathbf{x},t) = \tilde{u}_i(\tilde{\mathbf{x}},\tilde{t}), \tag{9}$$

which is the Galilean transformation of the fluctuating velocity.

We should note that the equations of motion for the mean velocity (the Reynolds equation) and the fluctuating velocity can be obtained by substituting the left hand side of Eq. (7) into Eq. (1) in place of V_i, and following the procedures leading to Eqs. (8) and (9), as just outlined. In this way it can be shown that the Reynolds equation is Galilean invariant and that the *Galilean invariance of the equation for the fluctuating velocity may be established by transforming the mean velocity only*.

If U_j is constant in space and time, the turbulence is spatially homogeneous. To emphasize that the mean velocity is now constant (in both space and time), we rename it as $U_j = K_j$. Then it is easily shown that every term of the Reynolds equation vanishes and that the equation for the fluctuating velocity takes the form

$$\frac{\partial u_i}{\partial t} + K_j \frac{\partial u_i}{\partial x_j} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu_0 \nabla^2 u_i. \tag{10}$$

We note that this equation is Galilean invariant under

$$K_i = c_i + \tilde{K}_i, \quad \frac{\partial \tilde{u}_i}{\partial t} = \frac{\partial \tilde{u}_i}{\partial \tilde{t}} - c_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}, \quad u_i = \tilde{u}_i, \tag{11}$$

as the two terms

$$\frac{\partial u_i}{\partial t} + K_j \frac{\partial u_i}{\partial x_j} \Rightarrow \frac{\partial \tilde{u}_i}{\partial \tilde{t}} - c_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + \tilde{K}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} + c_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}$$

transform invariantly due to the usual cancellation.

We further note that this also holds for $K_i = 0$, in which case the fluctuating velocities are measured in the comoving frame of the constant mean velocity. Thus, the equation for the fluctuating velocity in the comoving frame, viz.,

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu_0 \nabla^2 u_i, \tag{12}$$

is automatically Galilean invariant, for the special case where the mean velocity is a constant. That is, no actual test of Eq. (12) has to be made in order to establish its Galilean invariance. Nor is such a test possible without "accelerating" the field out of its comoving frame. This equation is the normal starting point for theoretical approaches and its Galilean invariance is erroneously taken as being identical to that of Eq. (1). Although the two equations are of identical form it must be borne in mind that Eq. (1) involves the instantaneous velocity whereas Eq. (12) involves the fluctuation from the mean.

In order to see the significance of this result for the RG, we introduce the Fourier transformation $f(\mathbf{k},t)$, for some arbitrary function $f(\mathbf{x},t)$. In order to establish the Galilean transformations for the Fourier transform, it is convenient to write the defining equation in $\tilde{S}$. Thus,

$$\tilde{f}(\tilde{\mathbf{k}},\tilde{t}) = \frac{1}{(2\pi)^3} \int e^{-i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} \tilde{f}(\tilde{\mathbf{x}},\tilde{t}) d^3\tilde{x}. \tag{13}$$

From this, and from Eq. (4) for the transformation of the differential coefficients, we have the transformation of wave number as $\mathbf{k} = \tilde{\mathbf{k}}$, so we will just use $\mathbf{k}$ for wave number in both frames.

Now we put $\tilde{\mathbf{k}} = \mathbf{k}$, $\tilde{t} = t$, and $d^3\tilde{x} = d^3x$ (the latter two, trivially) and substitute from Eq. (3) for $\tilde{x}$ into Eq. (13), to obtain

$$\tilde{f}(\mathbf{k},t) = \frac{1}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{c}t} \int e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{f}(\tilde{\mathbf{x}},t) d^3x. \tag{14}$$

This is the required result. In order to find the transformation rules for any $f(\mathbf{k},t)$, we simply substitute on the right-hand side for its real-space form in $\tilde{S}$.

Taking the velocity as an example, and substituting for $\tilde{V}_i(\tilde{\mathbf{x}},t)$ from Eq. (6), we have

$$\begin{aligned} \tilde{V}_i(\mathbf{k},t) &= e^{i\mathbf{k}\cdot\mathbf{c}t} \frac{1}{(2\pi)^3} \int e^{-i\mathbf{k}\cdot\mathbf{x}} \{V_i(\mathbf{x},t) - c_i\} d^3x \\ &= e^{i\mathbf{k}\cdot\mathbf{c}t} \{V_i(\mathbf{k},t) - c_i \delta(\mathbf{k})\}, \end{aligned} \tag{15}$$

and rearranging gives the reverse transformation as

$$V_i(\mathbf{k},t) = e^{-i\mathbf{k}\cdot\mathbf{c}t} \tilde{V}_i(\mathbf{k},t) + c_i \delta(\mathbf{k}). \tag{16}$$

Similarly, the direct and reverse transformations for the Fourier transform of the pressure take the form

$$\tilde{\Pi}(\mathbf{k},t) = e^{i\mathbf{k}\cdot\mathbf{c}t} \Pi(\mathbf{k},t), \quad \Pi(\mathbf{k},t) = e^{-i\mathbf{k}\cdot\mathbf{c}t} \tilde{\Pi}(\mathbf{k},t). \tag{17}$$

Now, Fourier transforming Eq. (1), the NSE takes the form in k space

$$\frac{\partial V_i(\mathbf{k},t)}{\partial t} + ik_j \int d^3l\, V_i(\mathbf{k}-\boldsymbol{l},t)V_j(\boldsymbol{l},t) = -ik_i\Pi(\mathbf{k},t) - \nu_0 k^2 V_i(\mathbf{k},t). \tag{18}$$

Substituting from Eqs. (16) and (17), we may write the left-hand side (LHS) of Eq. (18) as

$$\begin{aligned}[\text{LHS of Eq.}(18)] = &-i\mathbf{k}\cdot\mathbf{c}\widetilde{V}_i(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{c}t} + e^{-i\mathbf{k}\cdot\mathbf{c}t}\frac{\partial \widetilde{V}_i(\mathbf{k},t)}{\partial t}\\ &+ ik_j\int d^3l\, e^{-i\mathbf{k}\cdot\mathbf{c}t}\widetilde{V}_i(\mathbf{k}-\boldsymbol{l},t)\widetilde{V}_j(\boldsymbol{l},t)\\ &+ \underbrace{ik_j\int d^3l\, \widetilde{V}_i(\mathbf{k}-\boldsymbol{l},t)e^{-i(\mathbf{k}-\boldsymbol{l})\cdot\mathbf{c}t}c_j\delta(\boldsymbol{l})}_{A}\\ &+ \underbrace{ik_j\int d^3l\, \widetilde{V}_j(\boldsymbol{l},t)e^{-i\boldsymbol{l}\cdot\mathbf{c}t}c_i\delta(\mathbf{k}-\boldsymbol{l})}_{B}\\ &+ \underbrace{ik_j\int d^3l\, c_ic_j\delta(\mathbf{k}-\boldsymbol{l})\delta(\boldsymbol{l})}_{C}.\end{aligned} \tag{19}$$

Let us look at the terms labeled *A*, *B*, and *C* in turn. Clearly *B* and *C* vanish because of, respectively, incompressibility and the δ function at the origin. Term *A* cancels the first term on the right-hand side of Eq. (19), while the right-hand side of Eq. (18) may be transformed using Eqs. (16) and (17) so the Galilean transform of Eq. (18) takes the same form in variables with tildes.

Now we extend the Reynolds decomposition to wave number space by writing

$$V_i(\mathbf{k},t) = U_i(\mathbf{k},t) + u_i(\mathbf{k},t), \tag{20}$$

where $U_i(\mathbf{k},t)=\langle V_i(\mathbf{k},t)\rangle$ and $\langle u_i(\mathbf{k},t)\rangle=0$.

It follows at once from the Galilean transformations of Eqs. (8) and (10), and from Eq. (16), that the transformations for the mean and fluctuating velocities in *k* space are

$$U_i(\mathbf{k},t) = e^{-i\mathbf{k}\cdot\mathbf{c}t}\widetilde{U}_i(\mathbf{k},t) + c_i\delta(\mathbf{k}) \tag{21}$$

and

$$u_i(\mathbf{k},t) = e^{-i\mathbf{k}\cdot\mathbf{c}t}\widetilde{u}_i(\mathbf{k},t). \tag{22}$$

With the decomposition (20) substituted into Eq. (18) we can obtain equations of motion for the mean and fluctuating velocities in *k* space. However, we shall again concentrate on the case where the mean velocity is a constant and consider the equation of motion for the fluctuating velocity, where we have $U_i(\mathbf{k},t)=K_i\delta(\mathbf{k})$, and the transformation (21) becomes

$$K_i\delta(\mathbf{k}) = e^{-i\mathbf{k}\cdot\mathbf{c}t}\widetilde{K}_i\delta(\mathbf{k}) + c_i\delta(\mathbf{k}). \tag{23}$$

Fourier transforming the fluctuating pressure $p(\mathbf{x},t)$ in the same way as the velocity field, from Eq. (10), we have the *k*-space equation for the velocity fluctuation as

$$\frac{\partial u_i(\mathbf{k},t)}{\partial t} + ik_jK_ju_i(\mathbf{k},t) + ik_j\int d^3l\, u_i(\mathbf{k}-\boldsymbol{l},t)u_j(\boldsymbol{l},t) = -ik_ip(\mathbf{k},t) - \nu_0k^2u_i(\mathbf{k},t). \tag{24}$$

Galilean transformation of the right-hand side of Eq. (24) using Eqs. (17) and (22) is trivial so we concentrate on the left-hand side, where we also invoke Eq. (23), thus:

$$\begin{aligned}(\text{LHS}) = &\frac{\partial}{\partial t}[e^{-i\mathbf{k}\cdot\mathbf{c}t}\widetilde{u}_i(\mathbf{k},t)] + ik_j[\widetilde{K}_j + c_j]e^{-i\mathbf{k}\cdot\mathbf{c}t}\widetilde{u}_i(\mathbf{k},t)\\ &+ ik_j\int d^3l\, e^{-i(\mathbf{k}-\boldsymbol{l})\cdot\mathbf{c}t}\widetilde{u}_i(\mathbf{k}-\boldsymbol{l},t)e^{-i\boldsymbol{l}\cdot\mathbf{t}}\widetilde{u}_j(\boldsymbol{l},t).\end{aligned}$$

Differentiating the first square bracket generates $-i\mathbf{k}\cdot\mathbf{c}\widetilde{u}_i(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{c}t}$ which cancels with the term $ik_jc_j\widetilde{u}_i(\mathbf{k},t)e^{-i\mathbf{k}\cdot\mathbf{c}t}$ in the second square bracket. The exponentials in the last term reduce to the common factor $e^{-i\mathbf{k}\cdot\mathbf{c}t}$ which cancels the exponential factor on the right-hand side of Eq. (24). Hence the demonstration of Galilean invariance of the equation for the fluctuating velocity $u_i(\mathbf{k},t)$ relies, just as in *x* space, on the presence of the constant mean velocity K_i which is also transformed.

Now let us consider the effect on the RG of working in the laboratory frame rather than the usual comoving frame. For the RG, we are interested in the effect of filtering the equation of motion. Let us consider the Fourier modes $u(\mathbf{k},t)$ to be defined on the interval $0\leqslant k\leqslant k_0$. Then we define the filter as follows:

$$\begin{aligned}u^-(\mathbf{k},t) &= u(\mathbf{k},t), \quad 0\leqslant k\leqslant k_1;\\ u^+(\mathbf{k},t) &= u(\mathbf{k},t), \quad k_1\leqslant k\leqslant k_0.\end{aligned} \tag{25}$$

Here k_1 is the lower limit of the first wave number band for the RG.

Returning to the *k*-space equation of motion in the laboratory frame, as given by Eq. (24), we low-pass filter this and substitute the decomposition (25) into the nonlinear term (with an analogous decomposition for the pressure) to obtain

$$\begin{aligned}&\frac{\partial u_i^-(\mathbf{k},t)}{\partial t} + ik_jK_ju_i^-(\mathbf{k}) + ik_j\theta(k-k_1)\lambda_0\int d^3l\\ &\quad\times\{u_i^-(\mathbf{k}-\boldsymbol{l})u_j^-(\boldsymbol{l}) + u_i^-(\mathbf{k}-\boldsymbol{l})u_j^+(\boldsymbol{l}) + u_i^+(\mathbf{k}-\boldsymbol{l})u_j^-(\boldsymbol{l})\\ &\quad + u_i^+(\mathbf{k}-\boldsymbol{l})u_j^+(\boldsymbol{l})\}\\ &= -ik_ip^-(\mathbf{k}) - \nu_0k^2u_i^-(\mathbf{k}),\end{aligned} \tag{26}$$

where the function $\theta(k-k_1)$ satisfies

$$\theta(k-k_1) = \begin{cases}1 & \text{for } k\leqslant k_1,\\ 0 & \text{for } k > k_1.\end{cases}$$

Note that we have inserted a bookkeeping parameter $\lambda_0=1$ before the nonlinear term and that we have also omitted some time arguments in the interests of conciseness.

Now apply the Galilean transformation as given by Eq. (17) for the fluctuating pressure, Eq. (22) for the fluctuating velocity, and Eq. (23) for the constant translational velocity $\mathbf{K}$. It is immediately clear from a comparison with Eq. (18), and the way it is shown to be Galilean invariant using Eq.

(19), that Eq. (26) is similarly Galilean invariant. That is, the time derivative generates $-i\mathbf{k}\cdot\mathbf{c}u_i(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{c}t}$ which cancels the additive term in the Galilean transformation of the constant mean velocity **K**. The nonlinear terms generate phase factors $e^{-i\mathbf{k}\cdot\mathbf{c}t}$ which cancel right through the equation.

Next suppose that some operational procedure can be applied to the nonlinear terms involving the $\mathbf{u}^{+}(\mathbf{k},t)$ such that they can be replaced by renormalized viscous and vertex terms acting on the explicit scales $\mathbf{u}^{-}(\mathbf{k},t)$. In other words, we assume the existence of a conditional projector $\mathcal{P}_c$, such that

$$\mathcal{P}_c\{\mathbf{u}^{-}(\mathbf{k},t)\} = \mathbf{u}^{-}(\mathbf{k},t), \tag{27}$$

and with properties such that Eq. (26) takes the form

$$\frac{\partial u_i^{-}(\mathbf{k},t)}{\partial t} + ik_jK_ju_i^{-}(\mathbf{k},t) + ik_j\theta(k-k_1)\lambda(k) \times \int d^3l\, u_i^{-}(\mathbf{k}-\boldsymbol{l})u_j^{-}(\boldsymbol{l}) = -ik_ip^{-}(\mathbf{k}) - \nu(k)k^2u_i^{-}(\mathbf{k}), \tag{28}$$

where $\lambda(k)$ and $\nu(k)$ are renormalized vertex and viscous functions. In this picture, it is clear that the renormalizations of both vertex and viscosity are in fact unconstrained by Galilean invariance.

To sum up, as we pointed out after Eq. (12), theoretical papers in turbulence begin with what is invariably referred to as "the Navier-Stokes equation," but in fact this claim is not strictly true. Their starting point is really the equation for the fluctuating velocity in the comoving frame of the constant mean velocity. As we saw from Eqs. (10) and (11), in the laboratory frame the Galilean invariance of the equation for the fluctuating field is shown by transforming the constant mean velocity only. Hence the fluctuating field itself is free from constraint; and, as we showed symbolically in Eqs. (25)–(28), there is therefore no basis for constraining the perturbation expansion of the fluctuating velocity equation.

Indeed this conclusion could have been anticipated from elementary considerations. Vertex renormalization is to do with the dynamics and cannot be affected by kinematical shifts such as a Galilean transformation. Or, to put it another way, a velocity fluctuation is a velocity *difference* and so automatically Galilean invariant.

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